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COMMENT

Analytical inversion of a particular type of banded matrix

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Abstract. An alternative approach to that described in [1] is developed for analytically inverting a particular type of tridiagonal matrix. The technique is then extended to deal with a general banded matrix in which diagonal elements are identical and are flanked in each row by the same set of quantities.

A recent paper [1] developed an analytical approach to the inversion of a $k \times k$ tridiagonal matrix of the form

$$\mathbf{M} = \begin{pmatrix} D & 1 & 0 & 0 & \dots & & & & 0 \\ 1 & D & 1 & 0 & \dots & & & & 0 \\ 0 & 1 & D & 1 & \dots & & & & 0 \\ & & & & \dots & & & & \dots \\ & & & & \dots & & & & \\ & & & & \dots & & & & \dots \\ 0 & 0 & 0 & & \dots & 1 & D & 1 & 0 \\ 0 & 0 & 0 & & \dots & 0 & 1 & D & 1 \\ 0 & 0 & 0 & & \dots & 0 & 0 & 1 & D \end{pmatrix} \tag{1}$$

where D is an arbitrary constant. Their method was based on a calculation of the value of the determinant of the matrix together with its cofactors and led to an explicit expression for R_{pq} where $\mathbf{R} = \mathbf{M}^{-1}$. The purpose of the present contribution is to develop a simpler and more direct approach to the problem whose main advantage is that it may be applied to any $k \times k$ matrix with the following structure. The diagonal elements of the matrix are identical and are flanked in each row by the same set of $S(< k)$ non-zero elements, the remaining elements of the matrix being zero. We proceed to explain the technique by applying it first to the tridiagonal matrix $\mathbf{M}$ in (1) before showing how it may be generalized.

Since $\mathbf{MR} = \mathbf{I}$, it follows that

$$R_{p-1q} + DR_{pq} + R_{p+1q} = 0 \qquad (p \neq q) \tag{2}$$

$$R_{q-1q} + DR_{qq} + R_{q+1q} = 1 \qquad (p = q) \tag{3}$$

for $1 \leqslant p \leqslant k$ with $R_{0q} = R_{k+1q} = 0$. The structure of equation (2) is that of a homogeneous difference equation with constant coefficients and we therefore look for a solution of the form $R_{pq} = \text{constant} \times x^p$. This yields $x^2 + Dx + 1 = 0$ with solutions $x = \exp(\pm\lambda)$ where $\cosh\lambda = -D/2$. The general solution of equation (2) will therefore be of the form

$$R_{pq} = A_q \mathrm{e}^{p\lambda} + B_q \mathrm{e}^{-p\lambda} \tag{4}$$

but since equation (2) does not apply when $p = q$ we shall suppose equation (4) to hold for $p < q$, while for $p > q$ we take

$$R_{pq} = C_q e^{p\lambda} + E_q e^{-p\lambda}. \tag{5}$$

Here A_q, B_q, C_q, E_q are arbitrary constants whose value must now be determined. We begin with the 'boundary' conditions $R_{0q} = R_{k+1q} = 0$ given above, which relate the two constants present in each of equations (4) and (5) and allow these equations to be rewritten as

$$R_{pq} = F_q \sinh(p\lambda) \qquad (p < q) \tag{6a}$$

$$R_{pq} = G_q \sinh[(k+1-p)\lambda] \qquad (p > q). \tag{6b}$$

To determine F_q and G_q we note first that although equation (2) does not apply when $p = q$, the last term on the left-hand side involves R_{qq} when $p = q - 1$. Similarly, the first term on the left-hand side involves R_{qq} when $p = q + 1$. Thus for a consistent solution equations (6a) and (6b) must yield the same value for R_{qq} giving

$$R_{qq} = F_q \sinh(q\lambda) = G_q \sinh[(k+1-q)\lambda] \tag{7}$$

whence

$$F_q = H_q \sinh[(k+1-q)\lambda] \qquad G_q = H_q \sinh(q\lambda). \tag{8}$$

Finally we use equation (3) to determine H_q, choosing the relevant R_{pq} from equations (6) and (7). Hence we obtain

$$R_{pq} = \frac{-\sinh(p\lambda)\sinh[(k+1-q)\lambda]}{\sinh\lambda \sinh[(k+1)\lambda]} \qquad (p \leqslant q) \tag{9a}$$

$$R_{pq} = \frac{-\sinh[(k+1-p)\lambda]\sinh(q\lambda)}{\sinh\lambda \sinh[(k+1)\lambda]} \qquad (p \geqslant q) \tag{9b}$$

which is equivalent to the result given in [1].

We now proceed to consider the generalization of our technique to a $k \times k$ matrix $\mathbf{M}$ in which each of the (identical) diagonal elements a_0 is flanked on the left by the same set of $\alpha(\geqslant 0)$ non-zero elements $a_{-\alpha}, a_{-\alpha+1}, \ldots, a_{-1}$ and on the right by the same set of $\beta(\geqslant 0)$ non-zero elements $a_1, a_2, \ldots, a_\beta$. It is to be understood in this description that of necessity not all these elements will be present in the first α and the last β rows; thus, for example, the first row will be $a_0, a_1, \ldots, a_\beta, 0, 0 \ldots$ while the last row will be $\ldots, 0, 0, a_{-\alpha}, \ldots, a_{-1}, a_0$. If $\mathbf{R} = \mathbf{M}^{-1}$ it then follows that

$$\sum_{n=-\alpha}^{\beta} a_n R_{p+nq} = 0 \qquad (p \neq q) \tag{10a}$$

$$\sum_{n=-\alpha}^{\beta} a_n R_{q+nq} = 1 \qquad (p = q). \tag{10b}$$

These relations will clearly hold for $\alpha < p < k - \beta$ corresponding with those rows in which all the elements $a_{-\alpha}, \ldots, a_0, \ldots a_\beta$ occur. However, for those rows in which not all these elements are present, equations (10) will only hold if we arrange that the spurious terms occurring in the summations are zero. We can do this by supplementing equations (10) (now valid for $1 \leqslant p \leqslant k$) by the $S(= \alpha + \beta)$ 'boundary' conditions,

$$R_{1-\alpha q} = R_{2-\alpha q} = \ldots = R_{0q} = 0 = R_{k+1q} = R_{k+2q} = \ldots = R_{k+\beta q}. \tag{11}$$

We solve equation (10a) with a trial solution of the form $R_{pq} = \text{constant} \times x^p$ leading to

$$\sum_{m=0}^{S} a_{m-\alpha} x^m = 0 \tag{12}$$

with the set of solutions $x_1, x_2, \ldots, x_S$. This in turn yields the general solution for R_{pq} of the form

$$R_{pq} = \sum_{r=1}^{S} A_r x_r^p \qquad (p < q) \tag{13a}$$

$$R_{pq} = \sum_{r=1}^{S} C_r x_r^p \qquad (p > q) \tag{13b}$$

where the set of $2S$ constants A_r and C_r remain to be determined. It is to be understood here that A and C will (apart from r) also depend on k and q, but for simplicity we do not show this dependence explicitly. Now, the solution (13a) must satisfy equation (10a) for values of p up to and including $q-1$ and for this latter value terms appearing in (10a) include $R_{qq}, R_{q+1q}, \ldots R_{q+\beta-1q}$, all of which, except the first, are determined by equation (13b). Similarly the solution (13b) must satisfy equation (10a) for values of p down to and including $q+1$ and for this latter value, terms appearing in (10a) include $R_{qq}, R_{q-1q}, \ldots R_{q-\alpha+1q}$, all of which, except the first, are determined by equation (13a). For a consistent solution we therefore require that the values of $R_{q-\alpha+1q}, R_{q-\alpha+2q}, \ldots R_{qq} \ldots R_{q+\beta-1q}$ should be the same for equations (13a) and (13b), and this will yield a set of $S-1$ homogeneous linear relations between the A_r and C_r. The relations (11) (together with (13)) will yield a further set of S homogeneous linear relations between the A_r and C_r, while equation (10b) gives a single inhomogeneous relation between these quantities. We thus have a total of $2S$ linear equations which will determine the $2S$ constants A_r and $C_r (1 \leqslant r \leqslant S)$. The coefficients in these $2S$ equations will involve both k and q and an explicit solution of the equations by expansion of the relevant determinants (using Cramer's rule) can thus become (algebraically) prohibitive. For example, if $S = 5$ we would be involved with the non-trivial task of expanding a 10×10 determinant whose elements depend on k and q, and even if this were done, the complexity of the results would considerably detract from their value. We therefore proceed now to consider a technique for significantly simplifying the solution of these $2S$ linear equations. We shall show that the number of equations that require to be solved by Cramer's rule can be reduced to a number not exceeding $S/2$, so that in the above-mentioned case of $S = 5$, there will be at most two linear equations to be solved—a trivial task.

We begin with the $S-1$ relations arising from the requirement (given in the last paragraph) that equations (13a) and (b) should yield the same values for R_{pq} for $q-(\alpha-1) \leqslant p \leqslant q+(\beta-1)$. On defining

$$A'_r = A_r x_r^{q+1-\alpha} \qquad C'_r = C_r x_r^{q+1-\alpha} \tag{14}$$

these relations are readily expressed in the form

$$\sum_{r=1}^{S} (C'_r - A'_r) x_r^{\gamma} = 0 \qquad (0 \leqslant \gamma \leqslant S-2). \tag{15}$$

We now develop the left-hand side of equation (10b) by substituting for R_{q+nq} from equations (13). We use equation (13a) for $-\alpha \leqslant n \leqslant \beta-1$ and (13b) for $n = \beta$ (since

$n = \beta - 1$ is the largest value for which (13a) is valid). This gives

$$\sum_{n=-\alpha}^{\beta} a_n R_{q+nq} = \sum_{r=1}^{S} A_r x_r^q \left(\sum_{n=-\alpha}^{\beta-1} a_n x_r^n \right) + a_\beta \sum_{r=1}^{S} C_r x_r^{q+\beta}. \tag{16}$$

It follows from equations (12) that $\sum_{n=-\alpha}^{\beta-1} a_n x_r^n = -a_\beta x_r^\beta$ and substituting into equation (16) then yields

$$\sum_{n=-\alpha}^{\beta} a_n R_{q+nq} = a_\beta \sum_{r=1}^{S} (C_r - A_r) x_r^{q+\beta} = a_\beta \sum_{r=1}^{S} (C'_r - A'_r) x_r^{S-1}. \tag{17}$$

Equating this to unity allows equation (10b) to be taken together with equations (15) to give

$$\sum_{r=1}^{S} (C'_r - A'_r) x_r^\gamma = \Delta_{\gamma+1} \qquad (0 \leqslant \gamma \leqslant S-1) \tag{18a}$$

where $\Delta_{\gamma+1} = 0 (0 \leqslant \gamma \leqslant S-2)$ and $\Delta_S = 1/a_\beta$. Equations (18a) form a set of S equations for the S unknowns $C'_r - A'_r (1 \leqslant r \leqslant S)$ and may equivalently be written

$$\sum_{n=1}^{S} D_{mn} (C'_n - A'_n) = \Delta_m (1 \leqslant m \leqslant S) \tag{18b}$$

where

$$D_{mn} = x_n^{m-1}. \tag{18c}$$

D_{mn} is a $S \times S$ Vandermonde matrix [2] and it is known that its inverse is given by

$$\prod_{\substack{\sigma=1 \\ (\sigma \neq r)}}^{S} \frac{x - x_\sigma}{x_r - x_\sigma} = \sum_{t=1}^{S} [D^{-1}]_{rt} x^{t-1} \tag{19}$$

[2]; that is, $[D^{-1}]_{rt}$ is the coefficient of x^{t-1} in the expansion of the $S-1$ degree polynomial in x given by the left-hand side of equation (19) and characterized by a particular choice of x_r. Now, the solution of equation (18b) is

$$C'_r - A'_r = \sum_{t=1}^{S} [D^{-1}]_{rt} \Delta_t = \frac{[D^{-1}]_{rs}}{a_\beta} \tag{20}$$

since $\Delta_S (= 1/a_\beta)$ is the only non-zero component of Δ_t. Further, it follows from equation (19) that

$$[D^{-1}]_{rs} = \frac{1}{\prod_{\substack{\sigma=1 \\ (\sigma \neq r)}}^{S} (x_r - x_\sigma)} \tag{21}$$

and thus from equations (14) and (20) that

$$C_r - A_r = \left[a_\beta x_r^{q+1-\alpha} \prod_{\substack{\sigma=1 \\ (\sigma \neq r)}}^{S} (x_r - x_\sigma) \right]^{-1} (1 \leqslant r \leqslant S). \tag{22}$$

Equation (22) is an important result as it allows us to eliminate C_r in favour of A_r in equations (11) taken together with (13), and these will then yield a set of S equations for the S unknowns A_r. To progress further we shall now suppose that $\alpha \geqslant \beta$; our approach is easily modified if $\alpha < \beta$. Using equation (13a), the first set of relations in (11) may be rewritten as

$$\sum_{r=1}^{\alpha} D_{\nu r} A_r'' = -\sum_{r=\alpha+1}^{S} D_{\nu r} A_r'' \qquad (1 \leqslant \nu \leqslant \alpha) \tag{23}$$

where $D_{\nu r}$ is defined in equation (18c) and $A_r'' = A_r x_r^{1-\alpha}$. The second set of relations in (11) may be rewritten with the help of equations (13b) and (22) as

$$\sum_{r=1}^{S} A_r x_r^{k+\mu} = -\sum_{r=1}^{S} \Gamma_r x_r^{k+\mu} \qquad (1 \leqslant \mu \leqslant \beta) \tag{24}$$

where Γ_r is the right-hand side of equation (22). Now, since the inverse of the $\alpha \times \alpha$ Vandermonde matrix $D_{\nu r}$ is known (as discussed above), the set of α relations (23) may be used to express each A_r'' (and hence A_r) for $1 \leqslant r \leqslant \alpha$ as a linear combination of A_r'' (and hence A_r) for $\alpha + 1 \leqslant r \leqslant S$. These may then be used to eliminate A_r in equation (24) for $1 \leqslant r \leqslant \alpha$ in favour of A_r for $\alpha + 1 \leqslant r \leqslant S$. The net result of this will be to express equation (24) as a set of β equations for the β unknowns $A_r (\alpha + 1 \leqslant r \leqslant S)$, of the form

$$\sum_{r=\alpha+1}^{S} G_{\mu r} A_r = -\sum_{r=1}^{S} \Gamma_r x_r^{k+\mu} \qquad (1 \leqslant \mu \leqslant \beta) \tag{25}$$

and we note that since $\alpha \geqslant \beta$, the number of equations (25) is $\leqslant S/2$. When these equations have been solved for $A_r(\alpha + 1 \leqslant r \leqslant S)$, the values of $A_r(1 \leqslant r \leqslant \alpha)$ may be obtained from the already-calculated expressions for the latter in terms of the former. Finally the set of $C_r(1 \leqslant r \leqslant S)$ are obtained from equation (22). As regards the q dependence of A_r and C_r it follows from equation (22) that the only q dependence of the right-hand side of equation (25) is through a linear combination over t of terms x_t^{-q}. Thus in view of equations (23) and (22), A_r and C_r will take the forms $A_r = \sum_{t=1}^{S} B_{rt} x_t^{-q}$, $C_r = \sum_{t=1}^{S} E_{rt} x_t^{-q}$ where the $S \times S$ matrices B_{rt} and E_{rt} (with elements depending on k but independent of q) are given by implementing the procedures described above. It then follows from equations (13) and (22) that the general p, q dependence of R_{pq} will be of the form

$$R_{pq} = \sum_{r,t=1}^{S} B_{rt} x_r^p x_t^{-q} \qquad (p \leqslant q) \tag{26a}$$

$$R_{pq} = \sum_{r,t=1}^{S} E_{rt} x_r^p x_t^{-q} \qquad (p \geqslant q) \tag{26b}$$

where

$$E_{rt} = B_{rt} + x_r^q \Gamma_r \delta_{rt}.$$

Now, in solving equation (25) two approaches are possible, based on the observation that while the right-hand side depends on both k and q (since Γ_r depends on q), the matrix $G_{\mu r}$ will depend on k but will be independent of q. If, therefore, an expression is required for the elements of $\mathbf{R}$ in terms of both k and p and q, it will be necessary to solve equation (25) by Cramer's rule and direct expansion of the relevant determinants as outlined earlier, and for useful results (in the absence of a computer algebra program) this will be restricted to situations where the smaller of α and β does not exceed, say 4 or 5. However, if expressions are required for the elements of $\mathbf{R}$ in terms of p and q for a specified *numerical* value of k then expressions for $A_r(\alpha + 1 \leqslant r \leqslant S)$ may be obtained from equation (25) by first inverting $G_{\mu r}$ *numerically* before operating with this inverse on the q-dependent right-hand side. It is clear that this procedure can be effective for values of α or β significantly greater than the 4 or 5 mentioned above.

We now proceed to show that a considerable simplification occurs in the implementation of the above technique if we confine our attention to values of p and q which are well away from the borders of the matrix. By this we mean that we are interested in expressions for R_{pq} when the four quantities p, q, $k-p$, $k-q$ are all greater than Z where Z (to be considered in detail presently) depends on the roots $x_1, x_2, \ldots, x_S$, but is independent of k. This will mean that for sufficiently large values of k the vast majority of the k^2 values of R_{pq} will be given by the formulae which we will now derive. We begin by ordering the roots x_r, so that $|x_1| < |x_2| < \ldots < |x_S|$ and rewrite equation (13*a*) as

$$R_{pq} = \sum_{r=1}^{\alpha} A_r x_r^p + \sum_{r=\alpha+1}^{S} A_r x_r^p \qquad (p \leqslant q). \tag{27}$$

Now, as pointed out in the last paragraph, the first set of α conditions in (11) can be used to determine $A_r (1 \leqslant r \leqslant \alpha)$ in terms of $A_r (\alpha+1 \leqslant r \leqslant S)$, the relationship taking the form

$$A_r = \sum_{\mu=\alpha+1}^{S} F_{r\mu} A_\mu \qquad (1 \leqslant r \leqslant \alpha)$$

where the matrix $F_{r\mu}$ depends on the values of α and $x_1, \ldots, x_S$. This then allows equation (27) to be expressed in the form

$$R_{pq} = \sum_{r=\alpha+1}^{S} A_r \Big[x_r^p + \sum_{\mu=1}^{\alpha} F_{\mu r} x_\mu^p\Big] \qquad (p \leqslant q). \tag{28}$$

Now, since the xs are ordered such that $|x_r|$ for $\alpha+1 \leqslant r \leqslant S$ is greater than $|x_\mu|$ for $1 \leqslant \mu \leqslant \alpha$, it follows that for a sufficiently large value of p (let it be Z) $\sum_{\mu=1}^{\alpha} F_{\mu r} x_\mu^p$ can be neglected compared with x_r^p, and thus for $p > Z$ equation (13*a*) may be approximated by

$$R_{pq} = \sum_{r=\alpha+1}^{S} A_r x_r^p \qquad (p \leqslant q). \tag{29a}$$

Although the value of Z cannot be calculated exactly, it can be estimated from the inequality $(x_\alpha / x_{\alpha+1})^z \ll 1$ corresponding to $Z = \gamma / \ln(x_{\alpha+1}/x_\alpha)$ with γ lying in the interval of, say, 5–10. We now follow the same procedure with equation (13*b*), writing it initially in the form

$$R_{pq} = \sum_{r=1}^{\alpha} C_r x_r^p + \sum_{r=\alpha+1}^{S} C_r x_r^p \qquad (p \geqslant q). \tag{29b}$$

In this case the second set of β conditions in (11) may be used to eliminate the β values of $C_r (\alpha+1 \leqslant r \leqslant S)$ in favour of $C_r (1 \leqslant r \leqslant \alpha)$. Further, since the relevant conditions (11) are given at $p > k$ and we are interested in values of $p < k$, it follows by an argument similar to that used above that the proportional contribution to R_{pq} for those x_r with larger modulus will become smaller as p increases and hence if $k - p > Z$ the contribution to $\mathbf{R}$ arising from the second summation in (29*b*) can be neglected, so that equation (13*b*) can be approximated by

$$R_{pq} = \sum_{r=1}^{\alpha} C_r x_r^p \qquad (p \geqslant q). \tag{29c}$$

We now combine equations (29a) and (29c) with the result (22), but noting first that the latter was derived by equating expressions for R_{pq} when p is in the neighbourhood of $q, [q-(\alpha-1) \leqslant p \leqslant q+(\beta-1)]$, so that any conclusions are subject to the conditions $q > Z$ and $k-q > Z$. When these conditions are satisfied equations (29a) and (29c) imply that in equation (22) the only non-zero A_r corresponds to $\alpha+1 \leqslant r \leqslant S$, while the only non-zero C_r corresponds to $1 \leqslant r \leqslant \alpha$. Hence we obtain directly from equation (22) that for

$$1 \leqslant r \leqslant \alpha \qquad A_r = 0 \qquad \text{and} \qquad C_r = \left[a_\beta x_r^{q+1-\alpha} \prod_{\substack{\sigma=1 \\ (\sigma\neq r)}}^{S} (x_r - x_\sigma)\right]^{-1} \tag{30a}$$

while for

$$\alpha+1 \leqslant r \leqslant S \qquad A_r = -\left[a_\beta x_r^{q+1-\alpha} \prod_{\substack{\sigma=1 \\ (\sigma\neq r)}}^{S} (x_r - x_\sigma)\right]^{-1} \qquad \text{and} \qquad C_r = 0. \tag{30b}$$

Our final conclusion is thus that if the four quantities $p, q, k-p, k-q$ are all greater than Z, then

$$R_{pq} = -\left(\frac{1}{a_\beta}\right) \sum_{r=\alpha+1}^{S} x_r^{p-q+\alpha-1} \prod_{\substack{\sigma=1 \\ (\sigma\neq r)}}^{S} (x_r - x_\sigma)^{-1} \qquad (p \leqslant q) \tag{31a}$$

$$R_{pq} = \left(\frac{1}{a_\beta}\right) \sum_{r=1}^{\alpha} x_r^{p-q+\alpha-1} \prod_{\substack{\sigma=1 \\ (\sigma\neq r)}}^{S} (x_r - x_\sigma)^{-1} \qquad (p \geqslant q). \tag{31b}$$

We note that the form (31) corresponds to that given in equation (26) for the situation when $\mathbf{B}$ and $\mathbf{E}$ are both diagonal and independent of k. It is clear that if $k \gg Z$, then use of equations (31) to solve $\mathbf{M}\boldsymbol{y} = \boldsymbol{z}$ for $\boldsymbol{y}$ (given $\boldsymbol{z}$) will yield accurate values for y_p except for $1 \leqslant p \lesssim Z$ and $k - Z \lesssim p \leqslant k$.

A particular case for which explicit forms for R_{pq} can be found (for all k) is when $\mathbf{M}$ is a triangular matrix (upper or lower). Although the inverse of such a matrix can be found numerically by backward or forward substitution (respectively), such an approach is not designed to yield formulae of the type considered here, and we therefore proceed to illustrate our technique for a lower triangular matrix corresponding to $\alpha = S$ and $\beta = 0$. The first set of relations in (11) yield S homogeneous linear relations between the $A_r(1 \leqslant r \leqslant S)$ with the solution $A_r = 0(1 \leqslant r \leqslant S)$, while no constraints on the C_r arise from the second set of relations since $\beta = 0$. The values of C_r are thus given directly by equation (22) with $A_r = 0$ and lead to the result the $\mathbf{R}$ is a lower triangular matrix with elements given by

$$R_{pq} = 0 \qquad (p < q) \tag{32a}$$

$$R_{pq} = \left(\frac{1}{a_\beta}\right) \sum_{r=1}^{S} x_r^{p-q+\alpha-1} \prod_{\substack{\sigma=1 \\ (\sigma\neq r)}}^{S} (x_r - x_\sigma)^{-1} \qquad (p \geqslant q) \tag{32b}$$

for all values of k.

Finally, we give a numerical example of the implementation of the above procedures, taking, for the sake of simplicity, the case of $S = 3$ ($S = 2$ has already been illustrated earlier). Now in general, solutions x_r of equation (12) will have to be obtained numerically at the outset of the calculation and in order to avoid this and to derive simple formulae we shall take values for the a_n which give integer solutions for equation (12). We therefore consider the matrix

$$\mathbf{M} = \begin{pmatrix} -6 & 1 & 0 & 0 & 0 & \dots & & & & 0 & 0 \\ 11 & -6 & 1 & 0 & 0 & \dots & & & & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & \dots & & & & 0 & 0 \\ 0 & -6 & 11 & -6 & 1 & \dots & & & & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & -6 & 11 & -6 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & -6 & 11 & -6 & 1 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & -6 & 11 & -6 \end{pmatrix} \qquad (33)$$

corresponding to $\alpha = 2$, $\beta = 1$ with $a_{-2} = -6$, $a_{-1} = 11$, $a_0 = -6$, $a_1 = 1$. x then satisfies the equation $x^3 - 6x^2 + 11x - 6 = 0$ with solutions $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ so that

$$R_{pq} = A_1 + 2^p A_2 + 3^p A_3 \qquad (p \leqslant q) \qquad (34a)$$

$$R_{pq} = C_1 + 2^p C_2 + 3^p C_3 \qquad (p \geqslant q). \qquad (34b)$$

Equation (22) gives

$$C_1 - A_1 = \frac{1}{2} \qquad C_2 - A_2 = \frac{-1}{2^{q-1}} \qquad C_3 - A_3 = \frac{1}{(2 \times 3^{q-1})} \qquad (35)$$

while conditions (11) yield

$$A_1 + A_2 + A_3 = 0 \qquad (36a)$$

$$A_1 + \tfrac{1}{2} A_2 + \tfrac{1}{3} A_3 = 0 \qquad (36b)$$

$$C_1 + 2^{k+1} C_2 + 3^{k+1} C_3 = 0. \qquad (36c)$$

From equations (36c) and (35) we then obtain

$$A_1 + 2^{k+1} A_2 + 3^{k+1} A_3 = -\tfrac{1}{2} + 2^{k-q+2} - \tfrac{1}{2} \times 3^{k-q+2}. \qquad (36d)$$

Equation (36d) corresponds to equation (24) and together with equations (36a) and (36b) (corresponding with equation (23)) give three equations to determine the unknown A_1, A_2, A_3. Equations (36a) and (36b) readily yield $A_1 = \left(\frac{1}{3}\right) A_3$, $A_2 = \left(-\frac{4}{3}\right) A_3$ (as there are only two equations it is unnecessary to use properties of the Vandermonde matrix), and substituting into equation (36d) gives

$$A_3 = \left(-\tfrac{3}{2}\right)(3^{k-q+2} - 2^{k-q+3} + 1)/(3^{k+2} - 2^{k+3} + 1)$$

from which A_1, A_2, C_1, C_2, C_3 are readily found. Hence we obtain for

$$p \leqslant q \qquad R_{pq} = \frac{-(3^{k-q+2} - 2^{k-q+3} + 1)(3^{p+1} - 2^{p+2} + 1)}{2(3^{k+2} - 2^{k+3} + 1)} \qquad (37a)$$

$$p \geqslant q \qquad R_{pq} = R_{pq}(\text{for } p \leqslant q) + \tfrac{1}{2}(3^{p-q+1} - 2^{p-q+2} + 1) \qquad (37b)$$

and we note that the p, q dependence follows that given in equations (26). For the limiting forms of R_{pq} when p, q, $k-p$, $k-q$ are all greater than Z, we obtain from equations (31) that for

$$p \leqslant q \qquad R_{pq} = \frac{-3^{p-q+1}}{2} \qquad (38a)$$

$$p \geqslant q \qquad R_{pq} = \tfrac{1}{2} - 2^{p-q+1} \qquad (38b)$$

and these are readily seen to hold as the limiting forms of equations (37) for Z defined by $(\frac{2}{3})^Z \ll 1$, corresponding perhaps to $Z \sim 10$–15. Thus for $k \gtrsim 100$ the limiting forms (38) will give a very good approximation for the majority of the elements of $\mathbf{R}$.

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